Variation Operator

Dr. Rakesh K Kapania
Aerospace and Ocean Engineering Department
Virginia Polytechnic Institute and State University, Blacksburg, VA

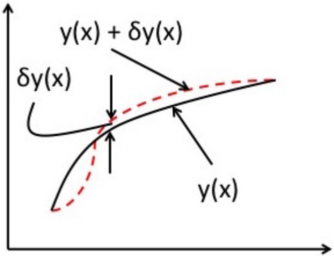
AOE 5024, Vehicle Structures
Summer, 2016

Euler-Lagrange Equations Using Delta Operator

Consider a Function $y(x)$ and a Neighborhood function $\tilde{y}(x)$ to the function $y(x)$

$$\tilde{y}(x) = y(x) + \delta y(x)$$

$\delta \Rightarrow$ Delta Operator or Variation Operator



Euler-Lagrange Equations Using Delta Operator

$\delta y(x) =$ An infinitesimal, slowly varying, change to the function at a given x. It vanishes at those points where $y(x)$ is specified.

$\delta y(x)$ is not same as $\dfrac{dy}{dx}$, it is $\delta(\dfrac{dy}{dx}) = \dfrac{d}{dx}(\delta y)$. It has the following properties:

- $\delta(F \pm G) = \delta F \pm \delta G$
- $\delta(FG) = F\delta G + G\delta F$
- $\delta\left(\dfrac{F}{G}\right) = \dfrac{G\delta F - F\delta G}{G^2}$

Euler-Lagrange Equations Using Delta Operator

Consider a Functional

$$I = \int_{x_1}^{x_2} F(x, y, y')dx$$

Here $y = y(x)$ (assumed to be continuous in $x_1 < x < x_2$), and y' indicates derivative of y with respect to x; and F, called the Lagrange function or Lagrangian, is a function of x, y and y'.

Objective: Determine $y(x)$ that will make I to be stationary.

$\delta y(x)$ vanishes at the boundary where the function $y(x)$ (essential boundary condition) is specified.

Euler-Lagrange Equations (contd...)

Variation of I

$$
\begin{aligned}
\delta I &= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right) dx \; + \; \text{higher order terms} \\
&= \delta^{(1)} I \; + \; \text{higher order terms}
\end{aligned}
$$

For I to have a stationary value, the first term on the right hand side, called the first variation of I, represented as $\delta^{(1)}$ must vanish ($i.e$ $\delta^{(1)} = 0$). This yields:

$$
\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right) dx = 0
$$

Euler-Lagrange Equations (contd...)

Integrating the second term by parts:

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'} \delta y'(x) dx = \frac{\partial F}{\partial y'} \delta y(x) \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \delta y(x) dx$$

Using the following Property of delta operator

$$\delta \left(\frac{dy}{dx} \right) = \frac{d\big(\delta y(x)\big)}{dx}$$

Stationary condition for I $\delta^{(1)} = 0$

$$\frac{\partial F}{\partial y'} \delta y(x) \Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \right] \delta y(x) dx = 0$$

Euler-Lagrange Equations (contd...)

$\delta y(x_1)$ and $\delta y(x_2)$ are arbitrary and independent

Above equation must be true for all values of $\delta y(x_1)$ and $\delta y(x_2)$ including when both are zero. This means the second term must be zero by itself.

$$\frac{\partial F}{\partial y'}\delta y(x)\bigg|_{x_1}^{x_2} = 0$$

$$\int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y(x)dx = 0$$

Since $\delta y(x)$ is an arbitrary and a slowly varying function of x

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Euler-Lagrange Equations (contd...)

Euler-Lagrange equation for the given Lagrangian, F.

$\delta y(x_1)$ and $\delta y(x_2)$ are independent from each other and are completely arbitrary

$$\left.\frac{\partial F}{\partial y'}\delta y(x)\right|_{x_1} = 0$$

$$\left.\frac{\partial F}{\partial y'}\delta y(x)\right|_{x_2} = 0$$

Either $\delta y(x_1) = 0$ ($y(x_1)$ specified), or $\dfrac{\partial F}{\partial y'} = 0$ at x_1,

and either $\delta y(x_2) = 0$ ($y(x_2)$ specified), or $\dfrac{\partial F}{\partial y'} = 0$ at x_2.

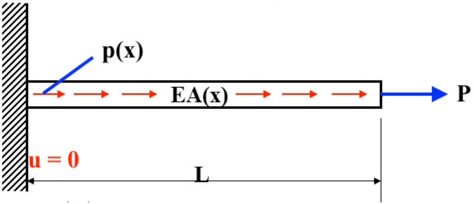
Euler-Lagrange Equations (contd...)

At a given end, $\dfrac{\partial F}{\partial y'}$ may be specified. For such cases the functional I will have additional term $G(y_1)$ or $(G(y_2))$.

For some cases, When $y(x)$ is specified at an end, we term it to be an **Essential Boundary Condition**

When $\dfrac{\partial F}{\partial y'} = 0$, it is called **Natural Boundary Condition**

Example



Total potential energy (Π)

$$\Pi = U + V$$

$U =$ Strain Energy of the bar

$$= \frac{1}{2}\int_0^L [EA(x)] \left(\frac{du}{dx}\right)^2 dx$$

$V =$ Potential of the applied load

$$= -\int_0^L p(x)u(x)dx - Pu(x = L)$$

Example (contd...)

Principle of minimum total potential energy

$\delta^{(1)}\Pi = 0$, $\delta^{(1)}\Pi$ is the first variation of the total potential energy

$$\delta^{(1)}\Pi = \frac{1}{2}\int_0^L [EA(x)] .2. \left(\frac{du}{dx}\right)\delta\left(\frac{du}{dx}\right)dx$$
$$- \int_0^L p(x)\delta u(x)dx - P\delta u(x = L)$$

By using integration by parts, and the fact that

$$\delta\left(\frac{du}{dx}\right) = \frac{d(\delta u)}{dx}$$

Example (contd...)

The first variation of Π becomes:

$$\begin{aligned} \delta^{(1)}\Pi &= EA(x)\left(\frac{du}{dx}\right)\delta u\Big|_0^L - \int_0^L \frac{d}{dx}\left[EA(x)\left(\frac{du}{dx}\right)\right]\delta u(x)dx \\ &- \int_0^L p(x)\delta u(x)dx - P\delta u(x=L) \end{aligned}$$

At $x = 0$, the displacement u is specified (an essential boundary condition). This renders $\delta u = 0$ at $x = 0$.

Example (contd...)

For Π to be stationary,

$$\left[EA(x)\left(\frac{du}{dx}\right)|_{x=L} - P\right]\delta u|_{x=L}$$
$$-\int_0^L \left[\frac{d}{dx}\{EA(x)\left(\frac{du}{dx}\right)\} + p(x)\right]\delta u(x)dx = 0$$

The above equation must be satisfied for all values of $\delta u(x = L)$, including $\delta u(x = L) = 0$

Euler Lagrange Equation

$$\frac{d}{dx}\left[EA(x)\left(\frac{du}{dx}\right)\right] + p(x) = 0$$

The associated boundary condition at $x = L$ is given as:

$$EA(x)\left(\frac{du}{dx}\right) = P$$

The boundary condition at $x = L$ is a **natural boundary condition**.

Lagrangian with Second Derivatives

Euler-Bernoulli Beam Strain energy contains the square of the second derivative of the transverse deflection

Euler-Lagrange equation will be a fourth-order ordinary differential equation with two boundary conditions at each end.

Consider

$$I = \int_{x_1}^{x_2} F(x, y, y', y'') dx$$

Here $y = y(x)$ (assumed to be continuous in $x_1 < x < x_2$), y' indicates derivative of $y(x)$ with respect to x; $''$ indicates second derivative of $y(x)$ with respect to x, and F, called the Lagrange function or Lagrangian, is a function of x, y, y' and y''

Objective:

Determine $y(x)$ that will make I to be stationary.

Lagrangian with Second Derivatives (contd...)

$$
\begin{aligned}
\delta I &= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' + \frac{\partial F}{\partial y''}\delta y'' \right) dx \\
&\quad + \text{higher order terms} \\
&= \delta^{(1)} I + \text{higher order terms}
\end{aligned}
$$

For I to have a stationary value $\delta^{(1)}$ must vanish *i.e.* $\delta^{(1)} = 0$.

$$
\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' + \frac{\partial F}{\partial y''}\delta y'' \right) dx = 0
$$

Lagrangian with Second Derivatives (contd...)

Integrating the second term by parts and using the fact that

$$\delta\left(\frac{dy}{dx}\right) = \frac{d(\delta y)}{dx}$$

yield:

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\delta y'(x)dx = \left.\frac{\partial F}{\partial y'}\delta y(x)\right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y(x)dx$$

Similarly, integrating the third term by parts and using the fact that

$$\delta\left(\frac{d^2y}{dx^2}\right) = \frac{d^2(\delta y)}{dx^2}$$

yield:

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y''}\delta y''(x)dx = \left.\frac{\partial F}{\partial y''}\delta y'(x)\right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\delta y'(x)dx$$

Lagrangian with Second Derivatives (contd...)

Performing the integration by parts one more time, we get:

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y''} \delta y''(x) dx = \frac{\partial F}{\partial y''} \delta y'(x) \Big|_{x_1}^{x_2} - \frac{d}{dx} \left(\frac{\partial F}{\partial y''} \right) \delta y(x) \Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial y''} \right) \delta y(x) dx$$

Combining:

$$\delta^{(1)} I = \frac{\partial F}{\partial y''} \delta y'(x) \Big|_{x_1}^{x_2} + \left[\frac{\partial F}{\partial y'} - \frac{d}{dx} \left(\frac{\partial F}{\partial y''} \right) \right] \delta y(x) \Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial y''} \right) \right] \delta y(x) dx$$

Lagrangian with Second Derivatives (contd...)

For I to be stationary, we want $\delta^{(1)} I = 0$.This yields:

$$\frac{\partial F}{\partial y''}\delta y'(x)\bigg|_{x_1}^{x_2} + \left[\frac{\partial F}{\partial y'} - \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\right]\delta y(x)\bigg|_{x_1}^{x_2}$$
$$+ \int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)\right]\delta y(x)dx = 0$$

Note that $\delta y(x)$, and $\delta y'(x)$ at a given end are arbitrary and independent of each other.

The above equation has to be satisfied for all possible values of $\delta y(x)$, and $\delta y'(x)$ at the two ends, including all four of them being zero *i.e.* $\delta y(x_1) = 0$, $\delta y(x_2) = 0$, $\delta y'(x_1) = 0$, and $\delta y'(x_2) = 0$.

Lagrangian with Second Derivatives (contd...)

This means:

$$\frac{\partial F}{\partial y''}\delta y'(x)\Big|_{x_1}^{x_2} = 0$$

$$\left[\frac{\partial F}{\partial y'} - \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\right]\delta y(x)\Big|_{x_1}^{x_2} = 0$$

$$\int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)\right]\delta y(x)dx = 0$$

Lagrangian with Second Derivatives (contd...)

From the First Equation, either

$$\delta y' = 0 \;\text{ or }\; \frac{\partial F}{\partial y''} = 0$$

First Condition

y is specified $\rightarrow$ Essential Boundary Condition

Second Condition $\rightarrow$ Natural or Force Boundary Condition

General Case

$\dfrac{\partial F}{\partial y''}$ may be specified and additional (boundary) term y' in the function

Lagrangian with Second Derivatives (contd...)

From the Second Equation, either

$$\delta y = 0 \ \text{ or } \ \left[\frac{\partial F}{\partial y'} - \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\right] = 0$$

First Condition

y is specified $\rightarrow$ Essential Boundary Condition

Second Condition $\rightarrow$ Natural or Force Boundary Condition

General Case

$\left[\frac{\partial F}{\partial y'} - \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\right]$ may be specified and additional (boundary) term y in the function

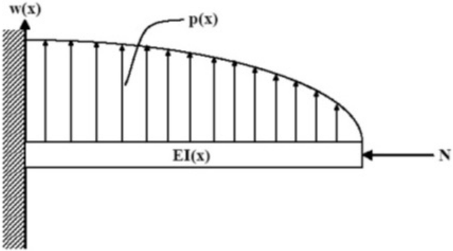
Lagrangian with Second Derivatives (contd...)

From the Third Term also being zero, the Euler-Lagrange equation is obtained by using the fact that variation $\delta y(x)$ is a slowly varying, arbitrary function of x.

$$\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)\right] = 0$$

The Euler-Lagrange equation for this case will be a fourth-order ordinary differential equation.

Example



The Total Potential energy of the system is:

$$\Pi = \frac{1}{2}\int_0^L \left[EI(x)\left(\frac{d^2w}{dx^2}\right)^2 - N\left(\frac{dw}{dx}\right)^2 \right] dx - \int_0^L p(x)w(x)dx$$

Determine the Euler-Lagrange equations along with the associated boundary conditions.

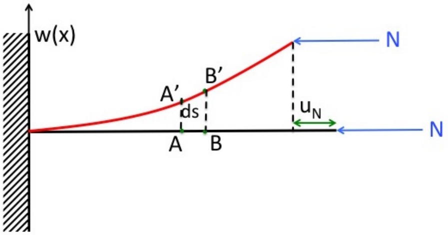
Example (contd...)

The **Strain Energy** of the system is,

$$U = \frac{1}{2} \int_0^L \left[EI(x) \left(\frac{d^2 w}{dx^2} \right)^2 \right] dx$$

The Potential of Compressive Load is,

Let $\mathbf{u_N}$ be the inward axial displacement of the right end due to transverse deflection.

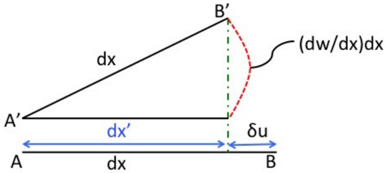


Example (contd...)

Now in order to determine u_N, zooming into the $'AB'$ section of the beam,

Here,

$$\delta u = dx - dx'$$



From the figure,

$$= \left[dx - \sqrt{dx^2 - \left(\frac{dw}{dx}\right)^2 dx^2} \right]$$

$$\simeq dx[1 - 1 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2] \simeq \frac{1}{2}\left(\frac{dw}{dx}\right)^2 dx$$

$$u_N = \frac{1}{2}\int_0^L \left(\frac{dw}{dx}\right)^2 dx$$

Example (contd...)

We have

$$F = \frac{1}{2}\left[EI(x)\left(\frac{d^2w}{dx^2}\right)^2 - N\left(\frac{dw}{dx}\right)^2\right] - p(x)w(x)$$

$$\frac{dF}{dw} = -p$$

$$\frac{dF}{dw'} = \frac{1}{2}.2(-N)\frac{dw}{dx}$$

$$\frac{dF}{dw''} = \frac{1}{2}.2EI(x)\frac{d^2w}{dx^2}$$

Example

The governing equation thus becomes:

$$-p(x) - \frac{d}{dx}\left[-N\frac{dw}{dx}\right] + \frac{d^2}{dx^2}\left[EI(x)\frac{d^2w}{dx^2}\right] = 0$$

This simplifies to:

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2w}{dx^2}\right] + \frac{d}{dx}\left[N\frac{dw}{dx}\right] = p(x)$$

Example (contd...)

The boundary conditions at $x = L$ are:

$$N\frac{dw}{dx} + \frac{d}{dx}\left[EI(x)\frac{d^2w}{dx^2}\right] = 0$$

$$EI(x)\frac{d^2w}{dx^2} = 0$$

Both the boundary conditions at $x = L$ are **natural boundary conditions**.
The boundary conditions at $x = 0$ are:

$$\begin{aligned} w &= 0 \\ \frac{dw}{dx} &= 0 \end{aligned}$$

The two boundary conditions at $x = 0$ are both **essential boundary conditions.**